

Group theory, coherent states, and the N-dimensional oscillator

C. R. Hagen*

*Department of Physics and Astronomy
University of Rochester
Rochester, N.Y. 14627-0171*

The isotropic harmonic oscillator in N dimensions is shown to have an underlying symmetry group $O(2,1) \times O(N)$ which implies a unique result for the energy spectrum of the system. Raising and lowering operators analogous to those of the one-dimensional oscillator are given for each value of the angular momentum parameter. This allows the construction of an infinite number of coherent states to be carried out. In the $N = 1$ case there is a twofold family of coherent states, a particular linear combination of which coincides with the single set already well known for that case. Wave functions are readily derived which require only the solution of a first order differential equation, an attribute generally characteristic of group theoretical approaches.

PACS numbers: 03.65.Pm; 03.65.-w

I INTRODUCTION

The importance of the simple harmonic oscillator to the study of quantum mechanical systems is virtually impossible to overstate. Its solution is well known both from the Schrödinger equation approach as well as the more elegant algebraic method based on raising and lowering operators. The extension to the case of N dimensions is quite trivial when formulated in terms of Cartesian coordinates since there is a total separation of variables in that case. The eigenvalues (eigenfunctions) are thus seen to be simply the sum (product) over the N individual oscillators. Somewhat less obvious is the analysis when spherical coordinates are employed. The usual route lies through the separation of radial and angular coordinates and a straightforward Schrödinger equation approach yields the energy eigenvalues as well as the wave functions. The latter are found to be expressible in term of Laguerre polynomials and generalized spherical harmonics. By contrast the goal here is to derive a solution which is entirely algebraic. Aside from the aesthetic appeal of such a representation independent approach, it will be seen to imply the existence of coherent states which have not heretofore been known to exist.

The starting point of this study is the Hamiltonian of the N -dimensional isotropic oscillator as given by

$$H = \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M\omega^2\mathbf{x}^2$$

where p_i and x_i with $i = 1, 2, \dots, N$ are respectively the canonical momenta and coordinates. These satisfy the commutation relations

$$[p_i, x_j] = -i\delta_{ij}$$

with $\hbar = 1$. The rescalings $x_i = Q_i(\sqrt{M\omega})^{-1}$, $p_i = P_i\sqrt{M\omega}$, and the definition $J_3 =$

* mail to hagen@pas.rochester.edu

$H/(2\omega)$ yield

$$J_3 = \frac{1}{4}(\mathbf{P}^2 + \mathbf{Q}^2).$$

Rotational invariance is invoked by defining the $\frac{1}{2}N(N-1)$ angular momentum operators $L_{ij} = Q_i P_j - Q_j P_i$ and the total angular momentum squared $L^2 = \frac{1}{2}L_{ij}L_{ij}$. In keeping with the avowed aim of this work to present the N -dimensional oscillator in a strictly group theoretical framework the reader is referred to [1] for the angular momentum aspect of the system. Using group theoretical methods the eigenvalues of L^2 have been shown there to be $\ell(\ell + N - 2)$ (a well known result) where ℓ is a nonnegative integer. Upon invoking the decomposition

$$L^2/\mathbf{Q}^2 = \mathbf{P}^2 - \mathbf{P} \cdot \mathbf{Q} \frac{1}{\mathbf{Q}^2} \mathbf{Q} \cdot \mathbf{P}$$

it follows that for a fixed L^2 eigenvalue J_3 takes the form

$$J_3 = \frac{1}{4}(\mathbf{P} \cdot \mathbf{Q} \frac{1}{\mathbf{Q}^2} \mathbf{Q} \cdot \mathbf{P} + \frac{\ell(\ell + N - 2)}{\mathbf{Q}^2} + \mathbf{Q}^2).$$

In the following section it is shown that the operator J_3 is an element of the algebra of the noncompact group $O(2,1)$ which then readily leads to a determination of the energy spectrum of the N -dimensional oscillator. Section **III** uses this algebra to construct a one parameter coherent state for each ℓ value. The wave functions of the system are subsequently derived in **IV** by direct application of the $O(2,1)$ framework. The exceptional case $N=1$ is discussed in section **V** with connection being made to well known coherent state results for that case. Section **VI** summarizes some of the main results.

II $O(2,1)$ ALGEBRA OF THE N -DIMENSIONAL OSCILLATOR

The demonstration of a $O(2,1)$ algebra requires the introduction of two additional operators, specifically

$$K_1 = -\frac{1}{4}[\mathbf{P} \cdot \mathbf{Q} \frac{1}{\mathbf{Q}^2} \mathbf{Q} \cdot \mathbf{P} + \frac{\ell(\ell + N - 2)}{\mathbf{Q}^2} - \mathbf{Q}^2]$$

and

$$K_2 = -\frac{1}{4}(\mathbf{P} \cdot \mathbf{Q} + \mathbf{Q} \cdot \mathbf{P}).$$

The set J_3, K_1 , and K_2 is closed under commutation; specifically

$$[J_3, K_1] = iK_2$$

$$[J_3, K_2] = -iK_1$$

$$[K_1, K_2] = -iJ_3$$

and thus provides a realization of the algebra of $O(2,1)$. Since the operators L_{ij} all commute with the set J_3 , K_1 , and K_2 , one has invariance with respect to the direct product $O(N) \times O(2,1)$. The Casimir operator of the $O(2,1)$ part of this algebra is defined by

$$C = J_3^2 - K_1^2 - K_2^2$$

and is found by direct calculation to be given by

$$C = \frac{1}{4}[(\ell + \frac{N}{2} - 1)^2 - 1].$$

It is convenient to define operators K_+ and K_- as given by

$$K_{\pm} \equiv K_1 \pm iK_2.$$

These respectively raise and lower by one unit the eigenvalues of J_3 as is evident from the commutation relations

$$[J_3, K_{\pm}] = \pm K_{\pm}.$$

Since repeated action of K_- on any eigenket of J_3 yields states of increasingly lower energy, it is clearly necessary by virtue of the positive definiteness of the Hamiltonian to require that for each value of ℓ there must be a state which is annihilated by K_- . In other words there must be some eigenvalue m_0 of J_3 such that

$$K_-|m_0, \ell\rangle = 0.$$

This is, of course, a straightforward adaptation of the technique used to solve for the spectrum of the one-dimensional oscillator. Rewriting the Casimir operator as

$$C = J_3^2 - J_3 - K_+K_-$$

is seen to imply that

$$C|m_0, \ell\rangle = (m_0^2 - m_0)|m_0, \ell\rangle.$$

This yields the quadratic equation

$$(m_0 - \frac{1}{2})^2 = \frac{1}{4}(\ell + \frac{N}{2} - 1)^2$$

which immediately gives the required eigenvalue m_0 as $\frac{\ell}{2} + \frac{N}{4}$ and/or $1 - \frac{\ell}{2} - \frac{N}{4}$. Upon application of K_+ to the state $|\ell, m_0\rangle$ a succession of n times ($n=0,1,2,\dots$) it is readily inferred that the energy spectrum $E_{n,\ell}$ is given by

$$E_{n,\ell} = \omega(2n + \ell + \frac{N}{2})$$

and/or

$$E_{n,\ell} = \omega(2n + 2 - \ell - \frac{N}{2}).$$

The former set of eigenenergies is the one which has been obtained in various contexts by separation of variables in the relevant Schrödinger equation and also by a partial group theoretical analysis of the Schrödinger equation [2]. Thus it is only the second set which requires detailed comment.

Since the energies of the system are necessarily positive, it is only the cases $N \leq 4$ of the second set which are possibly relevant. For $N = 4$ positivity requires $\ell = 0$ and it follows that the energy eigenvalue is zero. It implies a trivial realization of $O(2,1)$. Considering $N=2$ one finds that ℓ equal zero and one are both possible. The latter is also a trivial realization of $O(2,1)$ while the former corresponds to an eigenvalue already included in the other set of eigenvalues. This leaves only $N=3$ and 1 for discussion. If $N=3$ and $\ell=0$, one finds a ground state of $\frac{1}{2}\omega$, a value which must be excluded by the Cartesian coordinate solution for $N=3$. Finally, one is left only with the $N=1$ case and indeed this second set solution requires inclusion as it corresponds to the odd parity solution of the one-dimensional oscillator. One concludes that only the usual spectral values of the N -dimensional oscillator are relevant except for $N=1$. That special case is considered in section **V**.

III COHERENT STATES OF THE N-DIMENSIONAL OSCILLATOR

The construction of coherent states requires that the action of K_{\pm} on the energy eigenstates be determined. This can be done in close analogy to the case of the a and a^{\dagger} operators for the one dimensional oscillator. In particular one requires the evaluation of σ_n^{\pm} as defined by

$$K_{\pm}|n, \ell\rangle = \sigma_n^{\pm}|n \pm 1, \ell\rangle.$$

It is readily shown that

$$(\sigma_n^{\pm})^2 = \langle n, \ell | (J_3^2 \pm J_3 - C) | n, \ell \rangle$$

and thus

$$\sigma_n^+ = \sqrt{(n+1)(n+\alpha+1)}$$

and

$$\sigma_n^- = \sqrt{n(n+\alpha)}$$

where $\alpha \equiv \ell + \frac{N}{2} - 1$ and an undetermined phase has been absorbed into the definition of the states. It should be noted that this does not apply to the $N = 1$, $J_3 = \frac{3}{4}$ case as will be addressed in section **V**.

This is seen to allow the construction of K_- eigenstates $|k\rangle$ as

$$|k\rangle = \sum_{n=0}^{\infty} k^n B_n K_+^n |0, \ell\rangle$$

for some set B_n to be determined. Using the result for σ_n^{\pm} as given above implies that

$$|k\rangle = B_0 |0, \ell\rangle + \sum_{n=1}^{\infty} k^n B_n \prod_{i=1}^n [i(i+\alpha)]^{\frac{1}{2}} |n, \ell\rangle.$$

When K_- acts on this one obtains

$$K_-|k\rangle = k[B_1(1+\alpha)|0,\ell\rangle + \sum_{n=1}^{\infty} k^n B_{n+1} \prod_{i=1}^n [i(i+\alpha)]^{\frac{1}{2}} (n+1)(n+1+\alpha)|n,\ell\rangle]$$

so that $|k\rangle$ is an eigenvector of K_- with eigenvalue k provided that

$$B_n = B_{n+1}(n+1)(n+1+\alpha)$$

and

$$B_0 = B_1(1+\alpha).$$

These relations are satisfied with the choice

$$B_n = \frac{\Gamma(1+\alpha)}{n!\Gamma(n+1+\alpha)}$$

thereby establishing the existence of coherent states for the N -dimensional oscillator.

The time development of these states is obtained in the usual way by considering

$$|k\rangle_t \equiv \exp(-iHt)|k\rangle$$

readily yielding the result

$$|k\rangle_t = e^{-i\omega(\ell + \frac{N}{2})t} \sum_{n=0}^{\infty} (ke^{-2i\omega t})^n B_n K_+^n |0,\ell\rangle$$

thereby displaying the fact that the time development aside from a trivial phase factor consists solely in the replacement of k by the time dependent term $ke^{-2i\omega t}$. This result closely parallels the well known time development of the usual $N = 1$ coherent states. The norms of these states are readily computed from

$$\langle k|k\rangle = \langle 0| \sum_{n=0}^{\infty} k^{*n} K_-^n B_n^* \sum_{n'=0}^{\infty} k^{n'} K_+^{n'} B_{n'} |0\rangle.$$

This reduces to

$$\langle k|k\rangle = \sum_{n=0}^{\infty} (kk^*)^n |B_n|^2 \frac{n!\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}$$

whence it follows that

$$\langle k|k\rangle = \sum_{n=0}^{\infty} (kk^*)^n \frac{\Gamma(1+\alpha)}{n!\Gamma(n+1+\alpha)}.$$

More succinctly this is written as

$$\langle k|k\rangle = \Gamma(1+\alpha) I_{\alpha}(2|k|) |k|^{-\alpha}$$

where I_{α} denotes the usual modified Bessel function.

IV WAVE FUNCTIONS OF THE N-DIMENSIONAL OSCILLATOR

While the principal focus of this work is the algebraic aspect of the N -dimensional oscillator, it in fact provides a direct route to the wave functions of the system which requires only the solution of first order differential equations. One begins with the energy eigenvalues $E_{n\ell} = \omega(2n + \ell + \frac{N}{2})$, leaving to section V the exceptional case of $N=1$. The ground state satisfies the equation

$$K_-|0, \ell\rangle = 0$$

which in the coordinate representation implies

$$\langle r|(J_3 - \frac{1}{2}Q^2 - \frac{i}{2}Q \cdot P - \frac{N}{4})|0, \ell\rangle = 0.$$

Using the fact that the eigenvalue of J_3 for this state is $\frac{1}{2}(\ell + \frac{N}{2})$ it follows that

$$(\frac{1}{2}\ell - \frac{1}{2}r\frac{\partial}{\partial r} - \frac{1}{2}r^2)\langle r|0, \ell\rangle = 0$$

so that up to a normalization $\psi_{0\ell}(r) \equiv \langle r|0, \ell\rangle$ is given by

$$\psi_{0\ell}(r) = r^\ell e^{-r^2/2} L_0^{(\alpha)}(r^2).$$

In writing this equation use has been made of the Laguerre polynomial $L_n^{(\alpha)}(r^2)$ which is a constant for the case $n = 0$. It clearly anticipates the result for general n .

Making the ansatz that

$$\psi_{n\ell}(r) = A_n r^\ell e^{-r^2/2} L_n^{(\alpha)}(r^2)$$

the equation

$$K_+|n, \ell\rangle = \sigma_n^+|n+1, \ell\rangle$$

implies

$$-(n + \frac{\ell}{2} + \frac{N}{2} - \frac{1}{2}r^2 + \frac{1}{2}r\frac{\partial}{\partial r})A_n r^\ell e^{-r^2/2} L_n^{(\alpha)}(r^2) = \sigma_n^+ A_{n+1} r^\ell e^{-r^2/2} L_{n+1}^{(\alpha)}(r^2)$$

or with $r^2 = x$

$$-(n + \alpha + 1 - x + x\frac{\partial}{\partial x})A_n L_n^{(\alpha)}(x) = \sigma_n^+ A_{n+1} L_{n+1}^{(\alpha)}(x).$$

The Laguerre polynomials given by

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \left(\frac{\partial^n}{\partial x^n} \right) e^{-x} x^{n+\alpha}$$

are readily seen to be a solution of this equation for the coefficient choice

$$-(n+1)A_n = \sigma_n^+ A_{n+1}.$$

This allows one to write the normalized wave functions as

$$\psi_{n\ell}(r) = (-1)^n \left[\frac{2n!}{\Gamma(n + \alpha + 1)} \right]^{\frac{1}{2}} r^\ell e^{-r^2/2} L_n^{(\alpha)}(r^2)$$

in agreement with the usual result [3] save for the innocuous factor of $(-1)^n$. (The latter emerges from the fact that in the group theoretical approach employed here wave functions corresponding to different n values are related by successive applications of the K_+ operator.)

Wave functions of the coherent states are also readily derived. The result is expressible in the form

$$\langle r|k\rangle = r^{-N/2+1} e^{-r^2/2} [2\Gamma(1 + \alpha)]^{\frac{1}{2}} e^{-k} I_\alpha(2rk^{1/2}) k^{-\alpha/2}.$$

This yields the asymptotic behavior

$$\lim_{r \rightarrow \infty} r^{N-1} |\langle r|k\rangle_t|^2 = \frac{1}{2\pi} \Gamma(1 + \alpha) \exp[-(r - 2\sqrt{k} \cos \omega t)^2] e^{2k} k^{-(\alpha + \frac{1}{2})},$$

a result which closely resembles the corresponding result obtained in the $N = 1$ case for coherent states. The comparison is made even more striking by noting that in the limit $r, k \rightarrow \infty$ with $\sqrt{k} \ll r$ one has

$$\lim_{r, k \rightarrow \infty} \frac{r^{N-1} |\langle r|k\rangle_t|^2}{\langle k|k\rangle} = \frac{1}{\sqrt{\pi}} \exp[-(r - 2\sqrt{k} \cos \omega t)^2].$$

This result is in exact agreement with the normalized probability density which is well known to describe coherent states in the $N = 1$ case. It is worth noting that in the prescribed limits there is no longer any residual dependence on the angular momentum parameter ℓ .

V THE $N = 1$ CASE

It has been observed in this work that the $N = 1$ case is exceptional. In fact only for $N = 1$ are both of the energy eigenvalue sets $E = \omega(2n + \frac{1}{2})$ (even parity) and $E = \omega(2n + \frac{3}{2})$ (odd parity) allowed. For the former case the parameters σ_n^\pm are immediate as

$$\sigma_n^+ = \sqrt{(n+1)(n + \frac{1}{2})}$$

and

$$\sigma_n^- = \sqrt{n(n - \frac{1}{2})}.$$

For the odd parity case the corresponding results (here called $\tilde{\sigma}_n^\pm$) are readily found to be

$$\tilde{\sigma}_n^+ = \sqrt{(n+1)(n + \frac{3}{2})}$$

and

$$\tilde{\sigma}_n^- = \sqrt{n(n + \frac{1}{2})}.$$

One finds using these results that the wave functions for these two sets (in terms of the more appropriate coordinate x) are proportional to $e^{-x^2/2} L_n^{(-1/2)}(x^2)$ and $x e^{-x^2/2} L_n^{(1/2)}(x^2)$. As these Laguerre polynomials are the Hermite polynomials $H_{2n}(x)$ and $H_{2n+1}(x)$ respectively one finds the unified standard result for the wave functions of the one dimensional oscillator

$$\psi_n(x) = [2^n n!]^{-\frac{1}{2}} \pi^{-\frac{1}{4}} H_n(x) e^{-x^2/2}.$$

Coherent states for $N = 1$ are twofold. For even parity they follow from those previously derived by specializing to $\ell = 0$, $N = 1$. Specifically

$$|k\rangle = \sum_{n=0}^{\infty} k^n K_+^n \frac{\Gamma(\frac{1}{2})}{n! \Gamma(n + \frac{1}{2})} |0\rangle.$$

For the odd parity case the results for $\tilde{\sigma}_n^\pm$ yield the coherent state

$$|\tilde{k}\rangle = \sum_{n=0}^{\infty} \tilde{k}^n K_+^n \frac{\Gamma(\frac{3}{2})}{n! \Gamma(n + \frac{3}{2})} |\tilde{0}\rangle$$

where $\tilde{0}$ denotes the lowest odd parity state. The wave functions for these states are given by

$$\langle x|k\rangle = \pi^{-\frac{1}{4}} e^{-x^2/2} e^{-k} \cosh(2x\sqrt{k})$$

and

$$\langle x|\tilde{k}\rangle = \pi^{-\frac{1}{4}} e^{-x^2/2} e^{-\tilde{k}} \sinh(2x\sqrt{\tilde{k}})$$

respectively, and have the norms

$$\langle k|k\rangle = \Gamma(\frac{1}{2}) I_{-\frac{1}{2}}(2|k|) |k|^{\frac{1}{2}}$$

and

$$\langle \tilde{k}|\tilde{k}\rangle = \Gamma(\frac{1}{2}) I_{\frac{1}{2}}(2|\tilde{k}|) |\tilde{k}|^{\frac{1}{2}}.$$

More simply these are written as $\cosh(2k)$ and $\sinh(2\tilde{k})$ respectively. The implied time dependent probability densities are readily seen to be given by

$$\begin{aligned} |\langle x|k\rangle_t|^2 &= \frac{1}{4} \frac{1}{\sqrt{\pi}} e^{2k} |\exp[-\frac{1}{2}(x - 2\sqrt{k} \cos \omega t)^2] \exp(-2ix\sqrt{k} \sin \omega t) \\ &\quad + \exp[-\frac{1}{2}(x + 2\sqrt{k} \cos \omega t)^2] \exp(2ix\sqrt{k} \sin \omega t)|^2 \end{aligned}$$

and

$$\begin{aligned} \langle x|\tilde{k}\rangle|^2 = \frac{1}{4} \frac{1}{\sqrt{\pi}} e^{2\tilde{k}} & \exp\left[-\frac{1}{2}(x - 2\sqrt{\tilde{k}} \cos \omega t)^2\right] \exp(-2ix\sqrt{\tilde{k}} \sin \omega t) \\ & - \exp\left[-\frac{1}{2}(x + 2\sqrt{\tilde{k}} \cos \omega t)^2\right] \exp(2ix\sqrt{\tilde{k}} \sin \omega t)|^2. \end{aligned}$$

These are precisely what one would expect from the usual one-dimensional coherent state approach, given the fact that the development presented here effectively makes a separation into even and odd parity states.

In the usual formulation of the one-dimensional oscillator there is a single coherent state

$$|c\rangle = \exp(ca^\dagger)|0\rangle$$

where $a^\dagger = \frac{1}{\sqrt{2}}(Q - iP)$. It includes states of both parities, a fact that suggests the existence of an appropriate combination of the two coherent states found here (each of which contains states of only a given parity) that is equivalent to $|c\rangle$. Such a construction is facilitated by noting that

$$|\tilde{0}\rangle = a^\dagger|0\rangle.$$

Also, since $K^+ = \frac{1}{2}(a^\dagger)^2$ one writes $k = \tilde{k} = \frac{1}{2}c^2$ and seeks a realization of $|c\rangle$ of the form

$$|c\rangle = \sum_{n=0}^{\infty} \frac{c^{2n}}{2^{2n}n!} [(a^\dagger)^{2n} \frac{\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} + \lambda (a^\dagger)^{2n+1} \frac{\Gamma(\frac{3}{2})}{\Gamma(n + \frac{3}{2})}] |0\rangle$$

with λ to be determined. This relation is in fact verified with the specification $\lambda = c$, thereby completing the desired construction of $|c\rangle$ in terms of $|k\rangle$ and $|\tilde{k}\rangle$. As a final remark it should be noted that the two terms in the above equation are expressible as $\cosh(ca^\dagger)|0\rangle$ and $\sinh(ca^\dagger)|0\rangle$. Neither of these is an eigenvector of a despite the fact that they are individually eigenvectors of K_- by virtue of the fact that the latter is quadratic in a .

VI CONCLUSION

Despite the fact that the algebraic solutions of the one-dimensional and N-dimensional oscillators have long been known, there has not heretofore been a demonstration of the existence of such solutions which incorporate the spherical symmetry of the N-dimensional oscillator. The construction carried out in this work addresses that shortcoming, providing the first totally group theoretical solution of the N-dimensional oscillator. As a natural application of the results obtained it has been seen that wave functions are readily obtained by solving no more than a first order differential equation. Of most interest perhaps is the construction of coherent states of the N-dimensional oscillator. These have been shown to yield probability distributions which are asymptotically identical to those which have long been known to characterize the $N = 1$ case. In that latter application the coherent state results obtained have been shown to reproduce exactly well

known results which have been of great significance to the field of quantum optics. It will be of considerable interest to determine whether there exist domains of comparable applicability in the $N \neq 1$ case to other subfields of physics.

- [1] T. Friedmann and C. R. Hagen, J. Phys. Math. **53**, 122102 (2012).
- [2] H Bacry and J. L. Richard, J. Math. Phys. **8**, 2230 (1967).
- [3] R. Lynch and H.A. Mavromatis, J.Comp. and App. Math. **30**, 127 (1990).